A quantum annealing approach to the Minimum Multicut problem on general graphs

William Cruz-Santos[1] Salvador E. Venegas Andraca[2]
Marco Lanzagorta[3]

[1]Computer Engineering, CU-UAEM Valle de Chalco, Edo. de México, México

[2]Quantum Information Processing Group at Tecnológico de Monterrey, Escuela de Ciencias e Ingeniería

[3]US Naval Research Laboratory, 4555 Overlook Ave. SW Washington DC 20375, USA

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1 Introduction

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  Problem $\rightarrow$ QUBO $\rightarrow$ Embedding into the hardware

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- ◇ We discuss the limitations of the current family of quantum annealing processors.

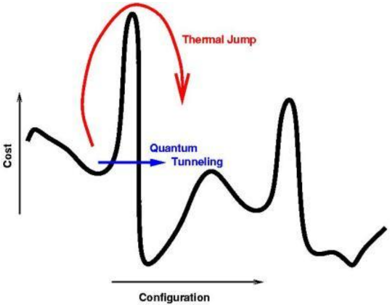
Contents

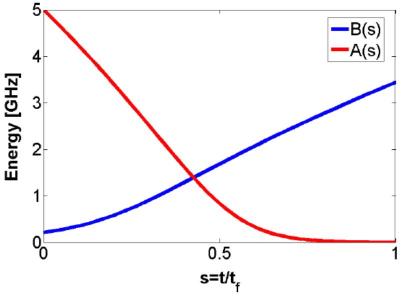
2 Quantum annealing

- QA annealing is used to travers from the ground state of an initial Hamiltonian to the ground state of the final Hamiltonian. [Finnila et al., 1994] [Kodawaki-Nishimori, 1998] [Farhi et al., 2001]

$$H(\tau) = A(s)H_I + B(s)H_{\text{problem}},$$

$$H_{\text{problem}} = \sum_{i}^{N} h_i \sigma_i^z + \sum_{j>i}^{N} J_{ij}\sigma_i^z\sigma_j^z, \qquad H_I = \sum_i \sigma_i^x$$





$$t_f = 20, \ldots, 2000\mu s$$

Adiabatic evolution

$$i\frac{d|\Psi(t)\rangle}{dt} = H(t)|\Psi(t)\rangle$$

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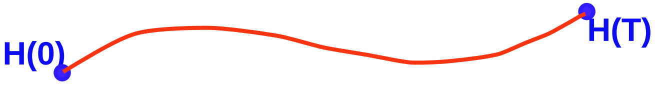
Adiabatic Theorem: [BornFock '28, Kato '51]



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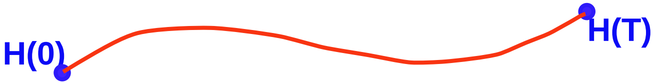

$|\Psi(0)\rangle$ Ground state of $H(0)$ $\longrightarrow$ $|\Psi(T)\rangle$ ground state of $H(T)$

$$T \gg \frac{1}{min_t\{\gamma(t)\}^2}, \quad \gamma = E_1(t) - E_0(t)$$

No crossing in the paths of the corresponding eigenvectors.

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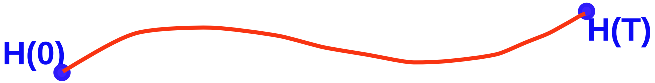

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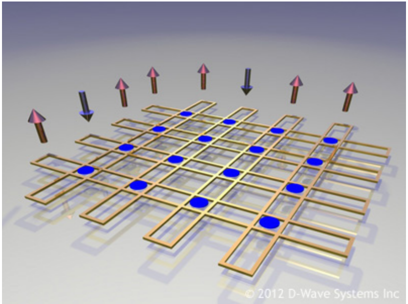
Linear interpolation between H_0 and H_1: [Farhi et al., 2001]

$$H(s) = (1-s)H_0 + sH_1, \quad s = \frac{t}{T}.$$

$$A(s) \sim (1-s), \quad B(s) \sim s$$

(Experimental) Quantum annealing



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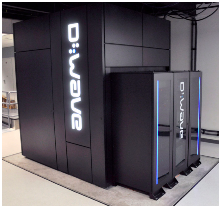

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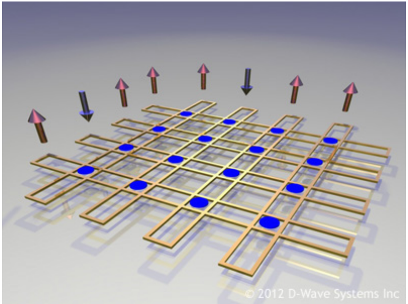

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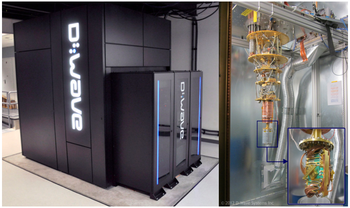

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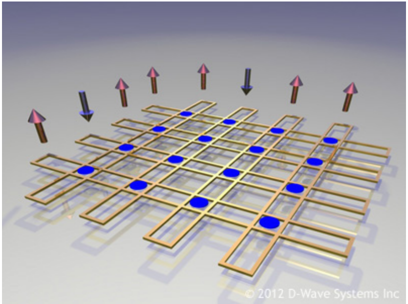

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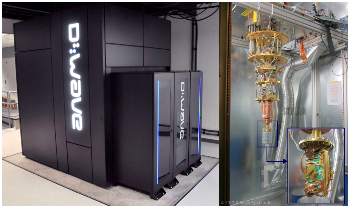

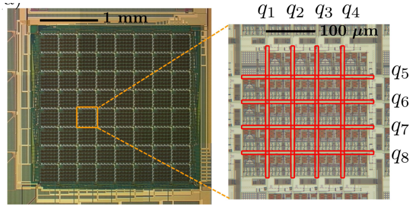

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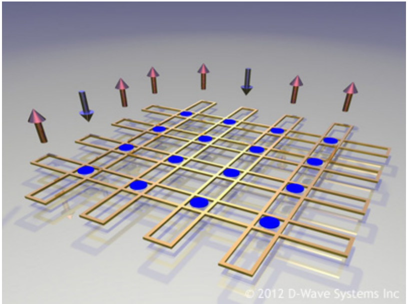

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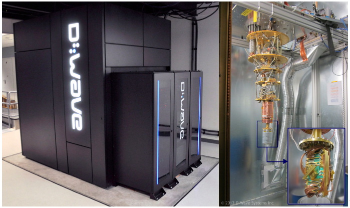

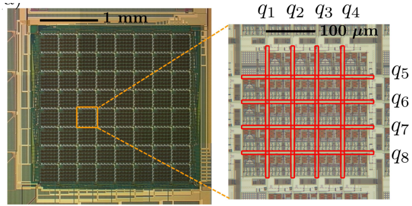

[Lanting et al, 2014]

Adiabatic quantum optimization

- The ground state of H_p corresponds to a configuration $\mathbf{s} = (s_1, \ldots, s_N) \in \{+1, -1\}^N$ of spins that minimize the following energy function

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From classical objective function to quantum Hamiltonian

Find the optimal assignment

$$\mathbf{s}^* = (s_1^*, \ldots, s_N^*)$$

$$E(\mathbf{s}) = \sum_{i}^{N} h_i s_i + \sum_{j>i}^{N} J_{ij} s_i s_j$$

$\Longleftrightarrow$

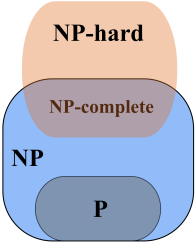
Find the ground state

$$|\psi_g\rangle = |\mathbf{s}^*\rangle = |s_1^*, \ldots, s_N^*\rangle$$

$$H_p = \sum_{i}^{N} h_i \sigma_i^z + \sum_{j>i}^{N} J_{ij} \sigma_i^z \sigma_j^z$$

3 Combinatorial optimization

- NPO is the class of optimization problems, NP-hard are the most difficult problems in NPO



- Factor ϵ-approximation algorithms $\mathcal{A}$ for problem Π,

$$\forall x \in \Pi \ : \ \mathsf{cost}_{\Pi}(x, \mathcal{A}(x)) \leq \epsilon \cdot \mathsf{OPT}(x).$$

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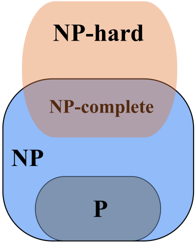

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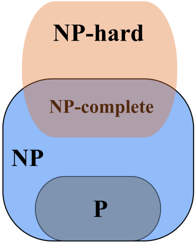

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The concept of inapproximated problems

Theorem [ALM, 1992]: There is a fixed $\epsilon > 0$ and a polynomial-time reduction τ from SAT to MAX-3SAT such that for every boolean formula I:

$$\begin{aligned} I \in \text{SAT} \quad &\Rightarrow \quad \text{MAX-3SAT}(\tau(I)) = 1 \\ I \notin \text{SAT} \quad &\Rightarrow \quad \text{MAX-3SAT}(\tau(I)) < \frac{1}{1+\epsilon}. \end{aligned}$$

In other words, achieving an approximation ratio $1 + \epsilon$ for MAX-3SAT is NP-hard.

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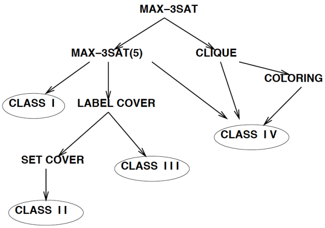
Classification of inapproximated problems [Arora-Lund, 1996]

Class	Representative problem	Hard ratio	Best ratio
I	MAX-3SAT	$1+\epsilon$	1.2987 [AHO+97]
	MULTIWAY CUTS		$3/2 - 1/\lvert S\rvert$ [CKR98]
II	MINIMUM SETCOVER	$O(\log n)$	$1 + \ln \lvert n\rvert$ [J97]
III	NEAREST LATTICE VECTOR	$2^{n \log^{1-\gamma}}$	Not in APX [ABS+97]
IV	MAXIMUM CLIQUE	n^{ϵ}	$O\left(\frac{n}{(\log n)^2}\right)$ [BH92]



4 Mapping of the Minimum multicut to QUBO

Minimum multicut: Given a weighted graph $G = (V, E, w)$ and a set of pairs $H = \{(s_1, t_1), \dots, (s_k, t_k)\} \subset V \times V$, find a multi-cut with minimum capacity, i.e., a subset $E' \subseteq E$ such that the removal of E' from E disconnects s_i from t_i for every pair (s_i, t_i), where the capacity of E' is given as $\sum_{e \in E'} w(e)$.

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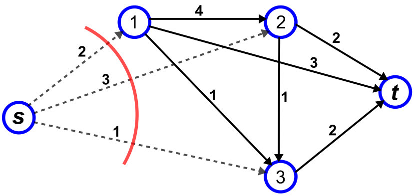

Min s-t cut

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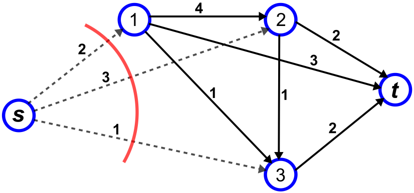

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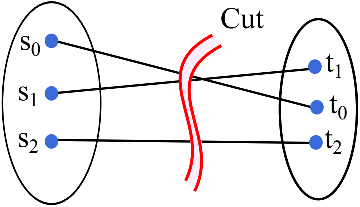

3-multicut

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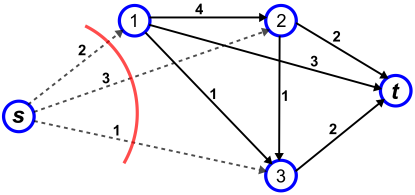

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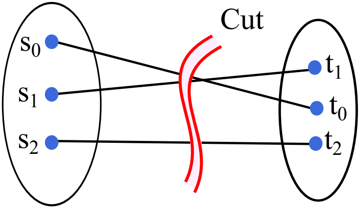


3-multicut

- For $k = 1, 2$, it is solvable in polynomial time. [Bollobas, 79] [Seymour, 79]
- For $k \geq 3$, Minimum Multi-Cut becomes APX-hard. [Dahlhaus, 94]
- It is NP-hard even if restricted to trees of height 1. [Garg et al., 97]

QUBO formulation of Minimum multicut in trees

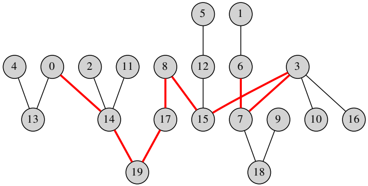
For each edge $e \in G$, $x_e = 1$ (in the cut), 0 (not in the cut)

$$h_G = h_{\text{weight}} + h_{\text{penalty}}$$

1. $h_{\text{weight}} = \sum_{e \in G} w(e)(1 - x_e)$
2. $h_{\text{penalty}} = \lambda_{\text{path}} \sum_{i=1}^{k} \prod_{e \in p_i} x_e$

   p_i is the path from s_i to t_i,

   $\lambda_{\text{path}} = \sum_{e \in p_i} w(e)$
3. $\deg(h_{\text{penalty}}) = \max_i \{\text{length}(p_i)\}$



There exists a unique path between every pair of vertices in a tree.

Reduction methods

$$f(x) = \sum_{S \subseteq [\![1,n]\!]} a_S \prod_{j \in S} x_j$$

$$\Downarrow \tau_r$$

$$f(x) = \min_{w \in \{0,1\}^m} g(x,w)$$

$$\deg\{g(x,w)\} \leq 2$$

w "ancilla variables"

τ_r "polynomial reduction"

(a) Negative terms can be reduced using only one extra ancilla variable [Freedman-Drineas, 2005]

$$-x_1 x_2 \cdots x_d = \min_{w \in \{0,1\}} w\left((d-1) - \sum_{j=1}^{d} x_j\right)$$

(b) For positive terms, only $\left\lfloor \frac{d-1}{2} \right\rfloor$ new ancilla variables are added.

$$\prod_{j=1}^{d} x_j = S_2 + \min_{w \in \{0,1\}^k} B - 2AS_1$$

if $d = 2k+2$,

$$\prod_{j=1}^{d} x_j = S_2 + \min_{w \in \{0,1\}^k} B - 2AS_1 + w_k(S_1 - d + 1)$$

if $d = 2k+1$.

See [Ishikawa, 2011].

(c) In the penalty approach, for each occurrence of xy, a new term is added. [Boros-Hammer, 2002]

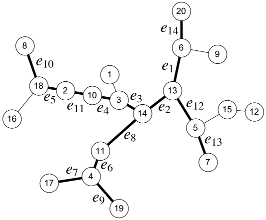
$$M(xy - 2xw - 2yw + 3w)$$

Upper bound: $M = 1 + 2 \sum_{S \subseteq [\![1,n]\!]} a_S$

Ancilla variables: $O(n^2 \log deg(f))$

Bad news: large coefficients

Example of reduction (1)

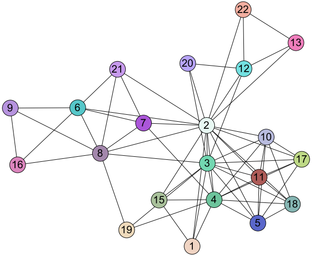


$$H = \{(6,10),(2,18),(11,17),(14,19),(8,13),(10,11),(3,5),(13,17),(7,14),(6,20)\}$$

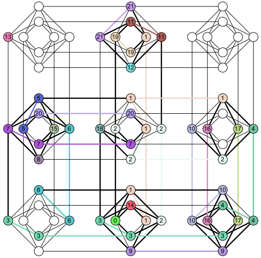
$$h_G = 14 - x_1 - x_2 - x_3 - x_4 + 9x_5 - x_6 - x_7 - x_8 - x_9 - x_{10} - x_{11} - x_{12} - x_{13} + 9x_{14} + 10x_1x_2x_3x_4 + 10x_6x_7 + 10x_6x_8x_9 + 10x_2x_3x_4x_5x_{10}x_{11} + 10x_3x_4x_8 + 10x_2x_3x_{12} + 10x_2x_6x_7x_8 + 10x_2x_{12}x_{13}$$

$$\Downarrow$$

h_G^{qubo}: 22 logical variables, 51 physical qubits

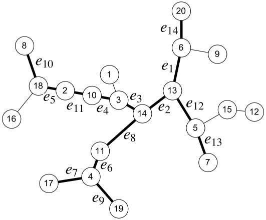


Logical graph of h_G^{qubo}.



Embedding into the Chimera.

Example of reduction (2)



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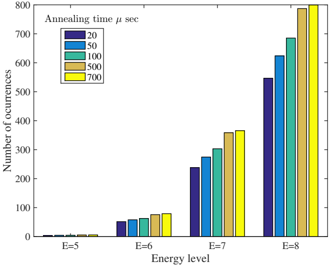
$$x_7 - x_8 - x_9 - x_{10} - x_{11} - x_{12} - x_{13} +$$
$$9x_{14} + 10x_1x_2x_3x_4 + 10x_6x_7 + 10x_6x_8x_9 +$$
$$10x_2x_3x_4x_5x_{10}x_{11} + 10x_3x_4x_8 + 10x_2x_3x_{12}$$
$$10x_2x_6x_7x_8 + 10x_2x_{12}x_{13}$$

$$\Downarrow$$

h_G^{qubo}: 22 logical variables, 51 physical qubits

Scalability of embedding

		logical variables	
n	k	H	H_{qubo}
20	3	10	17
30	5	14	23
45	6	22	37
100	30	75	199
100	130	97	402
100	200	99	559



Setup: $N_r = 100000$ readouts over 100 gauges.

QUBO formulation of Minimum multicut on general graphs

Given a graph $G = (V, E)$ and a set of pairs $H = \{(s_1, t_1), \ldots, (s_k, t_k)\}$. The Minimum multicut problem can be logically formulated as follows:

$$\min_{C \subseteq E} |C|. \bigwedge_{(s_i, t_i) \in H} \neg\text{connected}(s_i, t_i, C)$$

where

$$\text{connected}(s_i, t_i, C) \equiv \forall U \subseteq V. \varphi(s_i, t_i, C)$$

and

$$\begin{aligned} \varphi(s_i, t_i, C) \quad &\equiv \quad ((s_i \in U \wedge t_i \notin U) \rightarrow \\ &\quad\quad \exists x \in U. \exists y \notin U. \exists e \in E. inc(x, e) \wedge inc(y, e) \wedge e \notin C)). \end{aligned}$$

To verify if a given subset $C \subseteq E$ is a cut in G that disconnect every pair (s_i, t_i), then it is sufficient to find a subset $U \subseteq V$ such that $\neg\text{connected}(s_i, t_i, C)$ is true.

Mapping: Logical variables y_{uw} and x_v^i

- For each $\{u, w\} \in E$, $y_{uw} = 1$ (0) if $\{u, w\}$ is (not) selected for a cut.
- For each $v \in V$ and $i = 1, \ldots, k$, $x_v^i = 1$ (0) if v is (not) in U where U is a subset of V.

Construction: Let f_G be defined as

$$f_G = \text{card}(y_{uw}) + \alpha \cdot \text{penalty}(x_v, y_{uw}, H)$$

where

$$\text{card}(y_{uw}) = \sum_{\{u,w\}\in E} y_{uw} \quad \text{and}$$

$$\begin{aligned}\text{penalty} \quad &= \quad \sum_{i=1}^{k}\Big(\neg(x_{s_i}^i \oplus x_{t_i}^i) + \sum_{\{u,w\}\in E} (x_u^i \oplus x_w^i) \oplus y_{uw}\Big) \\ &= \quad \sum_{i=1}^{k}\Big(1 - x_{s_i}^i - x_{t_i}^i + 2x_{s_i}^i x_{t_i}^i + \\ &\qquad \sum_{\{u,w\}\in E} (x_u^i + x_w^i + y_{uw} - 2x_u^i x_w^i - 2x_u^i y_{uw} - \\ &\qquad 2x_w^i y_{uw} + 4x_u^i x_w^i y_{uw})\Big)\end{aligned}$$

Using the Ishikawa method we obtain

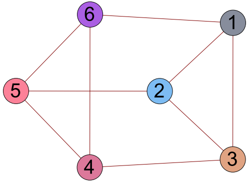
$$\begin{aligned}
\text{penalty} &= \sum_{i=1}^{k}\Big(1 - x^i_{s_i} - x^i_{t_i} + 2x^i_{s_i}x^i_{t_i} + \\
&\quad \sum_{\{u,w\}\in E} (x^i_u + x^i_w + y_{uw} - 2x^i_u x^i_w - 2x^i_u y_{uw} - \\
&\quad 2x^i_w y_{uw} + 4(x^i_u x^i_w + x^i_u y_{uw} + x^i_w y_{uw} + \\
&\quad z^i_{uw}(1 - x^i_u - x^i_w - y_{uw}))))\Big) \\
&= \sum_{i=1}^{k}\Big(1 - x^i_{s_i} - x^i_{t_i} + 2x^i_{s_i}x^i_{t_i} + \\
&\quad \sum_{\{u,w\}\in E} (x^i_u + x^i_w + y_{uw} + 2x^i_u x^i_w + 2x^i_u y_{uw} + 2x^i_w y_{uw} + \\
&\quad 4z^i_{uw}(1 - x^i_u - x^i_w - y_{uw}))\Big)
\end{aligned}$$

where z^i_{uw} are ancilla variables.

f_G uses $k(n+m)+m$ variables.

α is upper bounded by card(y_{uw})

Example of construction

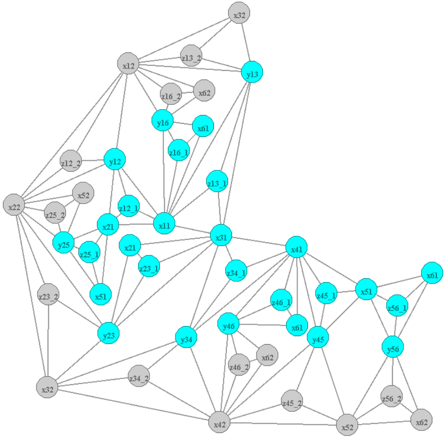


Boolean variables to represent the given problem:

$$x_1^1,\ x_2^1,\ x_3^1,\ x_4^1,\ x_5^1,\ x_6^1,\ x_1^2,\ x_2^2,\ x_3^2,\ x_4^2,\ x_5^2,\ x_6^2,\ y_{12},\ y_{13},\ y_{16},\ y_{23},\ y_{25},\ y_{34},\ y_{45},\ y_{46},\ y_{56}$$

Ancilla variables

$$z_{12}^1,\ z_{13}^1,\ z_{16}^1,\ z_{23}^1,\ z_{25}^1,\ z_{34}^1,\ z_{45}^1,\ z_{46}^1,\ z_{56}^1$$
$$z_{12}^2,\ z_{13}^2,\ z_{16}^2,\ z_{23}^2,\ z_{25}^2,\ z_{34}^2,\ z_{45}^2,\ z_{46}^2,\ z_{56}^2$$



Logical graph of f_G^{qubo}

5 Summary and conclusions

- ◇ The programming model is problem dependent.
- ◇ Can we avoid the reduction of pseudo-Boolean functions into QUBO?
- ◇ The minimum embedding is not always the best choice.
- ◇ Approximate solutions are also useful.
- ◇ To investigate programming inapproximated problems.

Thanks for your kind attention!

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Contact: wdelacruzd@uaemex.mx